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Fluctuations in a One-Dimensional Mechanical System. I. The Euler Limit

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We prove the central limit theorem for the density fluctuation field of a one-dimensional mechanical system (hard rods with equal masses and lengths and elastic collisions) in the hydrodynamic limit on the Euler time scale. The limiting process is deterministic and is governed by the linearized Euler equations of the model.

KEY WORDS: Nonequilibrium statistical mechanics; hydrodynamic limit; fluctuations; central limit theorem; Euler and Navier–Stokes equations; one-dimensional hard rods.

INTRODUCTION

The Euler equations describe the asymptotic behavior, on a certain scale in space-time, of the locally conserved fields of a classical system of Newtonian particles. In a suitable limit, called the hydrodynamic limit, the detailed description of the fluid in terms of the positions and velocities of a great number of particles goes over into a greatly simplified description in terms of a few continuous observables (mass density, momentum density, etc.). The mathematical reality behind this phenomenon is of course the law of large numbers (LLN), and the mathematical scenario for proving the validity of the Euler equations is by now well established.(5,6,17) Filling in the details is, however, extremely difficult, and a rigorous proof has been given only for certain models with simplifying features (ideal gases, one-dimensional harmonic oscillators, and certain models with stochastic dynamics). See the recent review in ref. 5.

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Probabilistically, the next level of description beyond the LLN is the central limit theorem (CLT). The CLT describes fluctuations, and a physical theory predicting the dynamics of these fluctuations around the Euler limit has also been worked out.[11,15] Essentially, the physical prediction is that the fluctuations evolve (on the same scale where the Euler equations are valid) deterministically, the dynamics being given by the *linearized* Euler equations linearized around the true solution (given the initial data). This reflects the fact that the fluctuations represent very mild perturbations away from the (nonrandom) value predicted by the Euler equation.

In this paper we prove an appropriate CLT for the fluctuation fields of a one-dimensional mechanical model, the so-called hard rods. These are one-dimensional hard cores of a fixed finite length, having equal masses, and moving at constant velocities between collisions. The collisions are assumed to be instantaneous and elastic. The hard rods are one of the very few models with nontrivial interactions for which the Euler equations have been derived rigorously[3,4] (a heuristic derivation was given earlier[18]). (In ref. 3 the authors state only the convergence of the expected value of the density field to the solution of the Euler equations, but we show that the LLN was implicit in their results; see Section 3.) The limiting fluctuation process is shown to agree with the physical prediction. For the limit of the time-correlation functions of the fluctuation field (in equilibrium), this result was obtained previously.[19]

The fluctuations can be thought of (probabilistically) as representing "corrections" to the LLN, but physically one expects "next corrections" to the Euler limit to involve space-time noise and dissipation. The corresponding hydrodynamic equations "with next corrections" are called the Navier–Stokes equations.

We have also investigated the appearance of these new phenomena, which occur on a longer time scale, in the fluctuation process. We find that on this new time scale the fluctuation process becomes non-Gaussian and stochastic. The covariance at several times determines a second-order operator which agrees with the predictions of Green–Kubo formulas.[19] We express this operator in terms of the Brownian motion of "pulses" in the hard rod fluid. These results will form the contents of a second paper.

The organization of this paper is as follows. In Section 1 we define the model and discuss space-time scalings and the Euler equation. In Section 2 we introduce the equilibrium fluctuation process and prove two forms of the CLT: a multivariate theorem for the fluctuations in occupation numbers of intervals at several times, and an "invariance principle" (Donsker's form of the CLT) for the density fluctuation field. In Section 3 we discuss the LLN and prove the first form of the CLT for the nonequilibrium case (in the same degree of generality as in ref. 3). As our CLT is derived from a theory of Billingsley concerning random sums of random variables, we present a short reprise of Billingsley's results in an appendix.

1. THE MODEL. SPACE-TIME SCALING AND THE EULER EQUATION

We first introduce the system of hard-rods and discuss its main properties. We give references to the literature rather than proofs for the facts claimed. We follow in the most part the notation of refs. 2–4.

The hard rods form an (infinite) mechanical system in one space dimension. Our rods will all have mass one and length $d>0$. A rod can have any velocity $v \in \mathbb{R}$. The phase space of the system of hard rods, which we denote by $\mathcal{M}_d$, is the set

$$\{(q_i, v_i): (q_i, v_i) \in \mathbb{R}^2,\\ q_i \leqslant q_{i+1} - d, -\infty < \cdots < q_{-1} < q_0 < q_1 < \cdots < +\infty\}$$

[Actually, we shall restrict the phase point (which we denote by X) to a subset $\mathcal{M}'_d \subset \mathcal{M}_d$ (in the notation of refs. 3 and 4), in which the dynamics is well defined.] A topology, a corresponding Borel σ-algebra, and an appropriate class of states (including Gibbs states) are defined on $\mathcal{M}_d$ in refs. 3 and 4. We confine ourselves to introducing a class of equilibrium states, each measure of which will be invariant under the hard-rod dynamics discussed below.

Choose a probability measure $h(dv)$ on $\mathbb{R}$ to play the role of velocity distribution. We assume that $h(\cdot)$ has finite moments (through fourth order), and that

$$\int_{-\infty}^{+\infty} v\, h(dv) = 0 \tag{1.1}$$

[If (1.1) is violated, we need only make a Galilean transformation.]

Choose a number $\rho < d^{-1}$ to play the role of the density. We define a measure P on $\mathcal{M}_d$ heuristically by declaring that P is translation invariant under space translations, that the interrod spacings are independent and exponentially distributed with mean $\rho^{-1}(1-\rho d)$, and that the velocities are independent of each other and of the rod positions and are distributed by $h(dv)$. More precisely, we define P by the following contraction procedure and Palm construction. Let (q_0, v_0) be a rod in X. Define $C_{q_0}(X)$ to be the point-particle phase point obtained from X by "contraction around q_0," i.e., $C_{q_0}(X) = \{(q'_i, v_i)\}$ with $q'_i = q_i - id$. Let P^0 be a Poisson point process on $\mathbb{R}^2$ with intensity $\rho_0\, dq\, h(dv)$, where $\rho_0 = (1-\rho d)^{-1}\rho$ is the "contracted density." Define $D_{(q_0, v_0)}$ so that $D_{(q_0, v_0)} C_{q_0} = \text{Identity}$, and let $P^{(q_0, v_0)} = P^0 \circ D^{-1}_{(q_0, v_0)}$. Then P is defined by setting, for any bounded, continuous function F on $\mathscr{M}'_d$,

$$\int F\, dP = \lim_{L\to\infty} (2L)^{-1} \int_{-L}^{L} \rho\, dq_0 \int h(dv_0) \int F\, dP^{(q_0, v_0)} \tag{1.2}$$

[Using the existence of finite moments,[3,4] it can be proved that $P(\mathscr{M}'_d) = 1$.]

We next define the hard-rod dynamics. It will be more convenient to follow the motion of a "pulse" rather than that of a given hard rod. Therefore, we allow the rods to interchange labels upon colliding. A collision thus affects a jump by $\pm d$ (instantaneously) in the motion of a pulse. The fluid can be regarded as a collection of "ideal gases," one for each velocity in the system, which interact with each other only by the jumps caused by collisions.

The motion of a single pulse is given by [let $(q_0, v_0) \in X$]

$$q(t) = q_0 + v_0 t + d n_X(q_0, v_0, t) \tag{1.3}$$

where $n_X(q_0, v_0, t)$ is the number of collisions suffered by the pulse during its motion up to time t. The latter can be computed in terms of the contracted picture as follows. Let T_t^0 denote the free (ideal-gas) evolution of a fluid of point particles. Then $n_X(q_0, v_0, t)$ is the number of crossings of the point $q(\tau) = q_0 + v_0 \tau$ by particles of velocity $v \neq v_0$ during the free evolution: $\tau \to T_\tau^0 C_{q_0} X$, $0 < \tau \leqslant t$. The evolution of the whole hard-rod configuration (which we denote by T_t) can also be given in terms of the free evolution (let S_b denote the shift in space by b):

$$T_\tau X = S_{n_X(q_0, v_0, t) d} D_{q_0 + \tau v_0} T_\tau^0 C_{q_0} X \tag{1.4}$$

$n_X(\cdot)$ is always well-defined if $X \in \mathscr{M}'_d$.[2-4]

One can also define in a similar fashion the motion of a "test pulse" added to the fluid at some instant t_0, defined to be a "zero-length hard-rod" which jumps by $\pm d$ at the instants of collisions with the true rods and otherwise moves with a preassigned velocity v_0. The only change needed to define its motion occurs if, at t_0, its initial position q_0 is overlapped by a rod. In that case we add to the definition of the collision number a "collision" occurring at time zero which moves q_0 instantaneously to the "outgoing" position.[3] This affects its motion by at most $\pm d$.

We next introduce hydrodynamic scalings and the locally conserved fields. $\varepsilon > 0$ will be our scaling parameter; $\varepsilon \to 0$ will define the "hydrodynamic limit." Only space and time will be rescaled. We will call the scaling $(q, t) \to (\varepsilon^{-1}q, \varepsilon^{-1}t)$, where (q, t) will now denote the macroscopic space-time point, the "Euler scaling" (it is the scaling under which the classical Euler equations remain invariant). Let $\varphi \in \mathscr{S}(\mathbb{R}^2)$ be a "test function" [$\mathscr{S}(\mathbb{R}^2)$ is Schwarz's space of smooth, rapidly decreasing functions], and let $N_X(dq, dv)$ be the configuration X, regarded as a locally finite measure on $\mathbb{R}^2$. Define

$$Z_t^\varepsilon(\varphi) \equiv \varepsilon \int N_{T_{\varepsilon^{-1}t}X}(\varepsilon^{-1}\, dq, dv)\, \varphi(q, v) \tag{1.5}$$

$Z_\cdot^\varepsilon(\cdot)$ is called the (rescaled) density field. Since velocities are individually conserved for hard rods, each field $Z_t^\varepsilon(\tilde{\varphi}\cdot\delta_v)$, $\tilde{\varphi} \in \mathscr{S}(\mathbb{R})$, $v \in \mathbb{R}$, is locally conserved, so that $Z_\cdot(\cdot)$ is a linear combination of locally conserved fields.

Hydrodynamic theory predicts that $Z_t^\varepsilon(\varphi)$ should have a deterministic limit, and in fact it was proven[3,4] that, for a suitable class of non-equilibrium initial states P^ε on $\mathscr{M}_d$ [$P^\varepsilon(\mathscr{M}'_d) = 1$ for all $\varepsilon > 0$],

$$\lim_{\varepsilon \to 0} E^\varepsilon Z_t^\varepsilon(\varphi) = Z_t(\varphi) \tag{1.6}$$

for any φ, t, and $\delta > 0$, where

$$Z_t(\varphi) = \iint dq\, dv\, \varphi(q, v)\, \rho(q, t; v) \tag{1.7}$$

is a process supported on continuous functions $\rho(\cdot, t; v)$ satisfying the Euler equation

$$\partial\rho(q, t; v)/\partial t = \partial/\partial q \left\{ v\rho(q, t; v) + d\rho(q, v, t) \right. \\ \left. \times \int dv'\, (v' - v)\, \rho(q, v'; t) \left[1 - d \int dv''\, \rho(q, v''; t) \right]^{-1} \right\} \tag{1.8}$$

Fluctuation theory concerns the CLT corrections to this deterministic limit.

2. EQUILIBRIUM FLUCTUATIONS ON THE EULER TIME SCALE

The Euler time scale refers, as mentioned above, to the space-time scaling $(\varepsilon^{-1}q, \varepsilon^{-1}t)$, $\varepsilon > 0$ and tending to zero. In this section we are interested in a limit theorem for the fluctuations of occupation numbers around their equilibrium values, with the Euler scaling, and jointly for several (macroscopic) times. In order to give the clearest exposition of our methods, we first give a limit theorem for the occupations of several intervals (a multivariate CLT), and then prove a stronger result for fluctuation fields, defined by integrating smooth test functions against the "fluctuation random measure" to be defined below. The latter is a kind of "invariance principle" for the fluctuation process. We first prove the CLT assuming that $h(dv)$ is a discrete measure supported on finitely many velocities $\{v_1,..., v_k\}$, i.e., we take $h(dv) = \sum_1^k h(v_i)\, \delta_{v_i}$. We show how to remove this restriction subsequently.

Let $N_t([a, b]; v)$ denote the number of hard rods of velocity v located in the interval $[a, b]$ at time t. We define the *fluctuation random measure* $Y^\varepsilon_\cdot(\cdot)$ by

$$Y^\varepsilon_t([a, b]; v) \equiv \varepsilon^{1/2}[N_{\varepsilon^{-1}t}([\varepsilon^{-1}a, \varepsilon^{-1}b]; v) - \rho h(v)\, \varepsilon^{-1}(b-a)] \quad (2.1)$$

Note that $Y^\varepsilon_\cdot(\cdot)$ has the Euler scaling of space and time, the correct centering, and a prefactor $\varepsilon^{1/2}$ (which anticipates normal fluctuations) built in. We shall prove a multivariate CLT for $(Y^\varepsilon_{t_i}([a_i, b_i]; v_i))_{i=1}^n$ for any choice of (t_i, a_i, b_i, v_i), $i = 1,..., n$, and compute the covariance structure of the limiting Gaussian variables.

Let $x_t(q, v)$ be the location at time zero of a velocity-v test pulse located at time t at q, and let $x^\varepsilon_t(q, v) \equiv x_{\varepsilon^{-1}t}(\varepsilon^{-1}q, v)$. Then we have the obvious equality

$$N_{\varepsilon^{-1}t}([\varepsilon^{-1}a, \varepsilon^{-1}b]; v) = N_0([x^\varepsilon_t(a, v), x^\varepsilon_t(b, v)]; v) + O(1) \quad (2.2)$$

where $O(1)$ is a term bounded by 1 in absolute value (independently of ε, a, b,...) which corrects for the possibility that one of the test pulses may be overlapped by a velocity-v rod at time t. Since $N_0(\cdot)$ is random, we have a random occupation of a random interval. Following the intuition coming from central limit theorems for random sums of random variables (see the Appendix), we rewrite (2.1) using (2.2) in order to obtain a term with the proper (random) centering:

$$\begin{aligned} Y^\varepsilon_t([a, b]; v) = \varepsilon^{1/2}\{&N_0([x^\varepsilon_t(a, v), x^\varepsilon_t(b, v)]; v) - \rho h(v)[x^\varepsilon_t(b, v) - x^\varepsilon_t(a, v)] \\ &+ \rho h(v)[x^\varepsilon_t(b, v) - x^\varepsilon_t(a, v) - \varepsilon^{-1}(b-a)]\} + O(\varepsilon^{1/2}) \quad (2.3) \end{aligned}$$

In (2.3) the first two terms in curly brackets form a quantity to which Billingsley's CLT for random sums applies, so it will have a Gaussian limit. We proceed to treat the remaining terms, which will fluctuate on the same scale. (The interaction with the "gases" of rods of velocities $w \neq v$ enters in these remaining terms.)

From the definition of the motion of a test pulse,

$$x_t^\varepsilon(q, v) = \varepsilon^{-1}q - \varepsilon^{-1}vt - dn_{\varepsilon^{-1}t}(\varepsilon^{-1}q, v) + O(1) \tag{2.4}$$

where $n_t(q, v)$ is the algebraic number of collisions of a test pulse of velocity v, located at time t at q, during the motion in the time interval $[0, t]$. [$n_t(q, v)$ is actually defined as the number of "collisions" (crossings) in the contracted picture, contracting about q at time t. Equation (2.4) and other relations below are all proved by using the contracted representation; for the details see discussion below.] We have

$$n_t(q, v) = \sum_{w \neq v} n_t(q, v; w) \tag{2.5}$$

where $n_t(q, v; w)$ is the number of collisions with rods of velocity w. The latter can easily be expressed in terms of time-zero occupation numbers:

$$n_t(q, v; w) = N_0([x_t(q, v), x_t(q, w)]; w) + O(1) \tag{2.6}$$

[Here and in the following we adopt the convention that $N([c, d]) \equiv -N([d, c])$ if it happens that $d < c$.] Equation (2.6) expresses in an obvious fashion the location at time zero of all the rods of velocity w which collided with our rod of velocity v during the motion.

Combining (2.4)–(2.6), we have

$$\begin{aligned} x_t^\varepsilon(b, v) - x_t^\varepsilon(a, v) &= \varepsilon^{-1}(b-a) - d \sum_{w \neq v} \{N_0([x_t^\varepsilon(a, w), x_t^\varepsilon(b, w)]; w) \\ &\quad - N_0([x_t^\varepsilon(a, v), x_t^\varepsilon(b, v)]; w)\} + O(1) \\ &= \varepsilon^{-1}(b-a) - d\varepsilon^{-1/2} \sum_{w \neq v} \{Y_0^\varepsilon([\varepsilon x_t^\varepsilon(a, w), \varepsilon x_t^\varepsilon(b, w)]; w) \\ &\quad - Y_0^\varepsilon([\varepsilon x_t^\varepsilon(a, v), \varepsilon x_t^\varepsilon(b, v)]; w)\} \\ &\quad - d\rho \sum_{w \neq v} h(w)\{x_t^\varepsilon(b, w) - x_t^\varepsilon(a, w) \\ &\quad - [x_t^\varepsilon(b, v) - x_t^\varepsilon(a, v)]\} + O(1) \end{aligned} \tag{2.7}$$

In (2.7) we have added and subtracted some terms in order to introduce some fluctuation numbers (with random intervals). Equation (2.7) can be viewed as an inhomogeneous system for the vector $\xi_v \equiv x_t^\varepsilon(b, v) - x_t^\varepsilon(a, v) - \varepsilon^{-1}(b-a)$ of form

$$(M\xi)_v = F_v \tag{2.8}$$

where

$$M = (1 - d\rho)I + d\rho P \tag{2.9}$$

P is the orthogonal projection onto 1 in $L^2(h(dv))$, and F_v denotes the terms containing $Y_0^\varepsilon(\cdot)$. Inverting, we obtain

$$\xi_v = \sum_w M_{v,w}^{-1} F_w \tag{2.10}$$

with

$$\begin{aligned} M^{-1} &= (1-d\rho)^{-1}(I - d\rho\, P) \\ &\equiv (1-d\rho)^{-1}\, C \end{aligned} \tag{2.11}$$

where we have introduced a matrix $C \equiv I - d\rho\, P$ for later convenience.

Substituting these results into (2.3), we arrive at

$$\begin{aligned} Y_t^\varepsilon([a,b];v) = {} & Y_0^\varepsilon([\varepsilon x_t^\varepsilon(a,v), \varepsilon x_t^\varepsilon(b,v)];v) \\ & - d\rho(1-d\rho)^{-1} h(v) \sum_w C_{v,w} \sum_{w'} \{Y_0^\varepsilon([\varepsilon x_t^\varepsilon(a,w'), \varepsilon x_t^\varepsilon(b,w')];w') \\ & - Y_0^\varepsilon([\varepsilon x_t^\varepsilon(a,w), \varepsilon x_t^\varepsilon(b,w)];w')\} + O(\varepsilon^{1/2}) \end{aligned} \tag{2.12}$$

The following facts will be proven later. For the asymptotic motion of the test pulses

$$\varepsilon x_t^\varepsilon(q,v) - q - \bar{v}t \xrightarrow[\varepsilon\to 0]{P} 0 \tag{2.13}$$

where $\bar{v} = (1-\rho d)^{-1} v$ is called the effective velocity of a pulse of (intrinsic) velocity v. For the asymptotics of the time-zero fields, let $Y_0([a,b];v)$ be a Gaussian random measure (i.e., for any set a_i, b_i, v_i, $i=1,\dots,n$, they are jointly Gaussian r.v.'s which are a.s. additive functions of the intervals) with covariance

$$EY_0([a,b];v)\, Y_0([c,d];w) = \rho h(v)\, C_{v,w}^2\, |[a,b] \cap [c,d]| \tag{2.14}$$

(C^2 is the square of the matrix C introduced earlier). Then

$$(Y_0^\varepsilon([a_i,b_i];v_i))_{i=1}^n \xrightarrow{d} (Y_0([a_i,b_i];v_i))_{i=1}^n \tag{2.15}$$

It follows immediately from (2.12)–(2.15) and Billingsley's theorem (see the Appendix) that

$$\begin{aligned} Y_t^\varepsilon([a,b];v) \xrightarrow[\varepsilon\to 0]{d} {} & Y_0([a-\bar{v}t, b-\bar{v}t];v) \\ & - d\rho(1-d\rho)^{-1} h(v) \sum_w C_{v,w} \sum_{w'} \{Y_0([a-\bar{w}'t, b-\bar{w}'t];w') \\ & - Y_0([a-\bar{w}t, b-\bar{w}t];w')\} \end{aligned} \tag{2.16}$$

jointly in distribution for any finite collection of a_i, b_i, v_i, and t_i, $i=1,\dots,n$.

Thus, the fluctuation random measures have a jointly Gaussian limit. The covariance structure of the limiting random measures can be computed from (2.16), but the formula simplifies considerably if we first introduce fields for the limiting process.

Let $\mathscr{S}(\mathbb{R}^2)$ be the Schwarz class of C^∞, rapidly decreasing "test functions" on $\mathbb{R}^2$, and let $\mathscr{S}'(\mathbb{R}^2)$ be its dual, the space of tempered distributions. Let P_0 be the probability measure on $\mathscr{S}'(\mathbb{R}^2)$ with characteristic function

$$E_{P_0} \exp[-iY_0(\varphi)] = \exp[-(1/2)(\varphi, C^2\varphi)_2] \tag{2.17}$$

where $Y_0(\varphi)(\xi) \equiv \langle \xi, \varphi \rangle_{s',s}$ and $(\ ,\ )_2$ denotes the inner product in $L^2(\rho h(dv)\, dq)$ (the existence and uniqueness of P_0 follow from Minlos' theorem[12]). P_0 is a Gaussian measure with covariance given by

$$\begin{aligned} E_{P_0} Y_0(\varphi)\, Y_0(\psi) &= (\varphi, C^2\psi)_2 \\ &= \sum_{v,w} \rho h(v)\, C^2_{v,w} \int dq\, \varphi(q, v)\, \psi(q, w) \end{aligned} \tag{2.18}$$

The random field $Y_0(\cdot)$ extends by taking limits in probability to "test functions" of form $1_{[a,b]}(q) \cdot \delta_v$, with covariance still given by (2.18).

As explained in the introduction, the hydrodynamic theory for equilibrium fluctuations predicts that the limiting Gaussian field will have the form

$$Y_t(\varphi) = Y_0([\exp(A_0^* t)]\, \varphi) \tag{2.19}$$

where A_0 is the generator of the linearized Euler flow [linearizing as $\rho h(v) + \delta f_t(q, v)\, h(v)$], and A_0^* is the adjoint of A_0 in $L^2(\rho h(dv)\, dq)$. {One expects to get A_0^* in (2.19) rather than A_0, since δf_t satisfies $(\partial/\partial t)\, \delta f_t = A_0\, \delta f_t$, and

$$\begin{aligned} Y_t(\varphi) &\sim \rho \int dq \sum_v h(v)\, \varphi(q, v)\, \delta f_t(q, v) \\ &= (\varphi, [\exp(A_0 t)]\, \delta f_0) = ([\exp(A_0^* t)]\, \varphi, \delta f_0) \\ &\sim Y_0([\exp(A_0^* t)]\, \varphi)\} \end{aligned}$$

In fact, one finds from (2.16) by explicit calculation for "test functions" of form $\varphi(q, v) = 1_{[a,b]}(q) \cdot \delta_v$,

$$Y_t(\varphi) = Y_0(T_t^* \varphi) \tag{2.20}$$

where

$$[T_t^* \varphi](q, v) = \sum_{w, w'} C_{w',w} C^{-1}_{v,w'}\, \varphi(q + \bar{w}' t, w) \tag{2.21}$$

It is easily checked that T_t^* is a semigroup with generator

$$A_0^* = -C^{-1} D_0 C \tag{2.22}$$

where

$$D_0 = \text{diag}(\bar{v}_{-k}, ..., \bar{v}_k)\, \partial/\partial q \tag{2.23}$$

Hence (since $C^* = C$, $D_0^* = -D_0$)

$$A_0 = C D_0 C^{-1} \tag{2.24}$$

and one checks without difficulty that A_0 is indeed the linearized Euler generator.

We summarize our first result in the following theorem (the restriction to discrete velocities is inessential; see below).

Theorem 2.1. Let $Y_t^\varepsilon(\cdot)$ be the rescaled fluctuation measure-valued process of the hard rods in equilibrium. Then $Y_t^\varepsilon(\cdot) \xrightarrow{d} Y_t(\cdot)$ (in the sense of convergence of finite-dimensional marginals), where $Y_t(\cdot)$ is a Gaussian measure-valued process satisfying

$$Y_t(\varphi) = Y_0([\exp(A_0^* t)]\, \varphi) \tag{2.25}$$

with A_0 the generator of the linearized Euler equations.

We next prove a stronger result for convergence of the random fields. Let P^ε be the distribution [on $D([0, \infty); \mathscr{S}'(\mathbb{R}^2))$, the space of paths lying in $\mathscr{S}'(\mathbb{R}^2)$, given its dual topology, which are right-continuous with left limits] of the distribution-valued process

$$Y_t^\varepsilon(\varphi) \equiv \varepsilon^{1/2} \left[\int N_{\varepsilon^{-1}t}(\varepsilon^{-1}\, dq, dv)\, \varphi(q, v) - \varepsilon^{-1} \rho \sum_v h(v) \int dq\, \varphi(q, v) \right] \tag{2.26}$$

Let P denote the process {also living in $D([0, \infty); \mathscr{S}'(\mathbb{R}^2))$, but in fact supported on $C([0, \infty); \mathscr{S}'(\mathbb{R}^2))$, the space of continuous paths} induced by $Y_t(\cdot)$. We shall prove that $P^\varepsilon \to P$ as $\varepsilon \to 0$ in the weak topology on measures. This means that, for every bounded, continuous functional F on path space, $E_{P^\varepsilon} F \to E_P F$. For example, one can take

$$F(T_\cdot) = \sup_{0 \leqslant \tau \leqslant t}\ \sup_{i=1,...,n} |\langle \xi_\tau, \varphi_i \rangle| \wedge 1, \qquad \varphi_i \in S, \quad i = 1,..., n$$

In order to establish this stronger form of convergence, we shall need the following facts, whose proof we postpone.

Lemma 2.1. Let P_0^ε (resp. P_0) be the distribution of $Y_0^\varepsilon(\cdot)$ [resp. of $Y_0(\cdot)$] on $\mathscr{S}'(\mathbb{R}^2)$. Then:

(i) $P_0^\varepsilon \to P_0$ weakly

(ii) $\sup_{\varepsilon>0} E\,|Y_0^\varepsilon(\varphi)|^2 \leqslant C\,\|\varphi\|_2^2$ for some constant C and all φ (2.27)

(iii) $EY_0^\varepsilon(\varphi)\,Y_0^\varepsilon(\psi) \to EY_0(\varphi)\,Y_0(\psi)$ for all φ, ψ (2.28)

Lemma 2.2. There is a Sobolev norm $|||\cdot|||$ on $\mathscr{S}(\mathbb{R}^2)$ and a constant C such that, for all φ, s, t,

$$\sup_{\varepsilon>0} E\,|Y_s^\varepsilon(\varphi) - Y_t^\varepsilon(\varphi)|^2 \leqslant C\,|||\varphi|||^2\,|s-t|^2 \tag{2.29}$$

Remark. The reader may have expected a term $O(|s-t|)$ on the right side of (2.29). The absence of this term reflects the deterministic nature of the evolution of the fluctuation fields on the Euler scale.

The convergence of finitely-many fields in distribution follows easily from Lemma 2.1, stationarity, and Theorem 2.1. In fact, for any φ_i, ψ_i, t_i, $i=1,\dots,n$ we have

$$\begin{aligned}
&\left|E\exp\left[i\sum\xi_j Y_{t_j}^\varepsilon(\varphi_j)\right] - E\exp\left[i\sum_j \xi_j Y_{t_j}^\varepsilon(\psi_j)\right]\right| \\
&\leqslant \frac{1}{2}E\left|\sum_{j=1}^n \xi_j Y_{t_j}^\varepsilon(\varphi_j-\psi_j)\right|^2 \\
&\leqslant \frac{n}{2}\sum_{j=1}^n \xi_j^2 EY_{t_j}^\varepsilon(\varphi_j-\psi_j)^2 \\
&= \frac{n}{2}\sum_{j=1}^n \xi_j^2 EY_0^\varepsilon(\varphi_j-\psi_j)^2 \\
&\leqslant \frac{cn}{2}\sum_{j=1}^n \xi_j^2\,\|\varphi_j-\psi_j\|_2^2
\end{aligned} \tag{2.30}$$

By approximating each $\varphi_i \in \mathscr{S}$ by linear combinations of functions of form $\delta_v \cdot 1_{[a,b]}(q)$ in L^2-norm, one can make the right side of (2.30) less than δ. Then letting $\varepsilon \to 0$, using Theorem 2.1, and letting $\delta \to 0$, one proves that finitely-many fields have a jointly Gaussian limit.

There remains only to prove tightness of the process. It suffices to establish the following two inequalities:

$$\sup_{\varepsilon>0} E\big[\sup_{0\leqslant\tau\leqslant t} |Y_\tau^\varepsilon(\varphi)|^2\big] \leqslant C'\,|||\varphi|||^2\,t^2 + C''\,\|\varphi\|_2^2 \tag{2.31}$$

$$\lim_{l\to 0}\sup_{\varepsilon>0} P\big[\sup_{\substack{0\leqslant\tau,\tau'\leqslant t\\ |\tau-\tau'|\leqslant l}} |Y_\tau^\varepsilon(\varphi) - Y_{\tau'}^\varepsilon(\varphi)| > \delta\big] \to 0 \tag{2.32}$$

for all t, φ, δ and some constants C' and C'' [$\|\cdot\|_2$ is the $L^2(\rho\, dq\, h(dv))$-norm].

Proof of (2.31). Let $\alpha = 2^{-n} \downarrow 0$. Since our process is separable,

$$
\begin{aligned}
&E[\sup_{0\leqslant\tau\leqslant t} |Y_\tau^\varepsilon(\varphi)|^2] \\
&= \lim_{\alpha\to 0} E\left[\sup_{0\leqslant\tau\leqslant t}\left|\sum_{k=1}^{[\tau\alpha^{-1}]} [Y_{k\alpha}^\varepsilon(\varphi) - Y_{(k-1)\alpha}^\varepsilon(\varphi)] + Y_0^\varepsilon(\varphi)\right|^2\right] \\
&\leqslant 2 \lim_{\alpha\to 0} [t\alpha^{-1}] E\left[\sum_{k=1}^{[t\alpha^{-1}]} |Y_{k\alpha}^\varepsilon(\varphi) - Y_{(k-1)\alpha}^\varepsilon(\varphi)|^2\right] + 2EY_0^\varepsilon(\varphi)^2 \\
&\leqslant \lim_{\alpha\to 0} [t\alpha^{-1}] \sum_{k=1}^{[t\alpha^{-1}]} C' \, |||\varphi|||^2 \, \alpha^2 + C'' \, \|\varphi\|_2^2 \\
&= C' \, |||\varphi|||^2 \, t^2 + C'' \, \|\varphi\|_2^2
\end{aligned}
\tag{2.33}
$$

where we used the Cauchy–Schwarz inequality in the second line and (2.27) in the third line.

Proof of (2.32).

$$
\begin{aligned}
&P[\sup_{|\tau-\tau'|\leqslant l} |Y_\tau^\varepsilon(\varphi) - Y_{\tau'}^\varepsilon(\varphi)| > \delta] \\
&= \lim_{\alpha\to 0} P\left\{\sup_{|\tau-\tau'|\leqslant l}\left|\sum_{[\tau\alpha^{-1}]}^{[\tau'\alpha^{-1}]} [Y_{k\alpha}^\varepsilon(\varphi) - Y_{(k-1)\alpha}^\varepsilon(\varphi)]\right| > \delta\right\} \\
&\leqslant \lim_{\alpha\to 0} P\left[\sup_{0\leqslant\tau\leqslant t} \sum_{[\tau\alpha^{-1}]}^{[(\tau+l)\alpha^{-1}]} |Y_{k\alpha}^\varepsilon(\varphi) - Y_{(k-1)\alpha}^\varepsilon(\varphi)| > \delta\right] \\
&\leqslant \lim_{\alpha\to 0} P\left\{\sup_{0\leqslant\tau\leqslant t} (l\alpha^{-1}+1)^{1/2}\left[\sum_{[\tau\alpha^{-1}]}^{[(\tau+l)\alpha^{-1}]} |Y_{k\alpha}^\varepsilon(\varphi) - Y_{(k-1)\alpha}^\varepsilon(\varphi)|^2\right]^{1/2} > \delta\right\} \\
&\leqslant \lim_{\alpha\to 0} P\left\{(l\alpha^{-1}+1)^{1/2}\left[\sum_{0}^{[(t+l)\alpha^{-1}]} |Y_{k\alpha}^\varepsilon(\varphi) - Y_{(k-1)\alpha}^\varepsilon(\varphi)|^2\right]^{1/2} > \delta\right\} \\
&\leqslant \lim_{\alpha\to 0} \delta^{-2}(l\alpha^{-1}+1) \sum_{0}^{[(t+l)\alpha^{-1}]} E\,|Y_{k\alpha}^\varepsilon(\varphi) - Y_{(k-1)\alpha}^\varepsilon(\varphi)|^2 \\
&\leqslant \lim_{\alpha\to 0} \delta^{-2}(l\alpha^{-1}+1)[(t+l)\,\alpha^{-1}+1]\,C\,|||\varphi|||^2\,\alpha^2 \\
&= \delta^{-2} l(l+t)\,C\,|||\varphi|||^2
\end{aligned}
\tag{2.34}
$$

where the Cauchy–Schwarz inequality, the Chebyshev inequality, and (2.27) have been used in lines four, six, and seven, respectively.

We summarize in the following theorem (the last statement will be proven in Section 4).

Theorem 2.2. The fluctuation field $Y_t^\varepsilon(\varphi)$ converges weakly to the Gaussian (generalized) Ornstein–Uhlenbeck process[14] $Y_t(\varphi)$ satisfying

$$dY_t(\varphi) = Y_t(A_0^*\varphi)\,dt \tag{2.35}$$

with initial condition

$$Y_0(\varphi) = Y_{(0)}(\varphi) \tag{2.36}$$

where $Y_{(0)}(\cdot)$ is Gaussian with covariance

$$EY_{(0)}(\varphi)\,Y_{(0)}(\psi) = (\varphi, C^2\psi)_2 \tag{2.37}$$

In addition, for all φ, ψ, $t \geqslant 0$,

$$\lim_{\varepsilon \to 0} EY_t^\varepsilon(\varphi)\,Y_0^\varepsilon(\psi) = EY_t(\varphi)\,Y_0(\psi) = (\varphi, e^{A_0 t}C^2\psi)_2 \tag{2.38}$$

We now give the proofs of the facts used in the previous analysis. We begin with the asymptotics of the time-zero field.

We use the contracted representation, contracting around the origin. We consider first the case that the origin is uncovered {i.e., we consider the conditional hard-rod state, conditioned on $X \cap [-d, 0) = \varnothing$}. The contracted state is Poisson with density $\rho_0 = \rho(1-\rho d)^{-1}$. Let $a < b$ be given and let $C_0^\varepsilon(a)$, $C_0^\varepsilon(b)$ be the (random) contracted positions of $\varepsilon^{-1}a$, $\varepsilon^{-1}b$ (resp.). These are given by, for $q > 0$,

$$C_0^\varepsilon(q) = \sup\{r : r + dN_0^c([0, r]) < \varepsilon^{-1}q\} \tag{2.39}$$

where $N_0^c([c, d])$ is the occupation number of $[c, d]$ in the contracted point process. In terms of the contracted configuration the hard-rod fluctuation measure is given by

$$\begin{aligned} Y_0^\varepsilon([a, b]; v) &= \varepsilon^{1/2}\{N_0^c([C_0^\varepsilon(a), C_0^\varepsilon(b)], v) - \rho h(v)\,\varepsilon^{-1}(b-a)\} \\ &= \varepsilon^{1/2}\{N_0^c([C_0^\varepsilon(a), C_0^\varepsilon(b)], v) - \rho_0 h(v)[C_0^\varepsilon(b) - C_0^\varepsilon(a)] \\ &\quad + \rho_0 h(v)[C_0^\varepsilon(b) - C_0^\varepsilon(a)] - \varepsilon^{-1}(1-\rho d)(b-a)\} + O(\varepsilon^{1/2}) \end{aligned} \tag{2.40}$$

Using (2.39), we obtain

$$\begin{aligned} Y_0^\varepsilon([a, b]; v) = \varepsilon^{1/2}(&N_0^c([C_0^\varepsilon(a), C_0^\varepsilon(b)]; v) - \rho_0 h(v)[C_0^\varepsilon(b) - C_0^\varepsilon(a)] \\ &- \rho\, dh(v) \sum_w \{N_0^c([C_0^\varepsilon(a), C_0^\varepsilon(b)]; w) \\ &- \rho_0 h(w)[C_0^\varepsilon(b) - C_0^\varepsilon(a)]\} + O(\varepsilon^{1/2}) \end{aligned} \tag{2.41}$$

Let $Y_0^\varepsilon(\cdot)$ be a Gaussian random measure with covariance

$$EY_0^\varepsilon([a,b];v)\,Y_0^\varepsilon([c,d];w)=\rho_0 h(v)\,\delta_{v,w}\,|[a,b]\cap[c,d]| \quad (2.42)$$

By an easy argument from (2.39), we obtain a weak law of large numbers for contractions:

$$\varepsilon C_0^\varepsilon(q)-(1-d\rho)q \xrightarrow{P} 0 \quad (2.43)$$

In fact, if $q>0$, $\delta>0$, $\hat{\delta}=\delta-\varepsilon d$,

$$\begin{aligned} &P[\varepsilon C_0^\varepsilon(q)>(1-d\rho)q+\delta] \\ &\quad =P[C_0^\varepsilon(q)>[(1-d\rho)q+\delta]\,\varepsilon^{-1}, \varepsilon^{-1}q-dN_0^\varepsilon([0,C_0^\varepsilon(q)]) \\ &\quad >[(1-d\rho)q+\hat{\delta}]\,\varepsilon^{-1}] \\ &\quad \leqslant P[dN_0^\varepsilon([0,(1-d\rho)\,q\varepsilon^{-1}+\hat{\delta}\varepsilon^{-1}])<d\rho\, q\varepsilon^{-1}-\hat{\delta}\varepsilon^{-1}] \\ &\quad =P[-\xi>d^{-1}(1-d\rho)^{-1}\,\hat{\delta}\varepsilon^{-1}] \end{aligned} \quad (2.44)$$

where ξ is the centered Poisson of variance $(\rho q+\rho_0\delta)\,\varepsilon^{-1}$. Therefore, by Chebyshev's inequality the last line is less than $d^2(1-d\rho)^2\,\delta^{-2}(\rho q+\rho_0\delta)\varepsilon$, which tends to 0. A similar argument applies to $P[\varepsilon C_0^\varepsilon(q)<(1-d\rho)q-\delta]$.

Using Billingley's theorem (see the Appendix), we obtain

$$Y_0^\varepsilon([a,b];v) \xrightarrow{d} Y_0^\varepsilon([\bar{a},\bar{b}];v)-\rho d\sum_w Y_0^\varepsilon([\bar{a},\bar{b}];w), \qquad \bar{q}\equiv(1-\rho d)q \quad (2.45)$$

For the general case we can disintegrate the hard-rod equilibrium state μ as $\mu=\mu_\phi+\int_{-d}^0 d\theta\,\mu_\theta$, where μ_ϕ is the previous state and μ_θ is μ conditioned to θ being the first uncovered point on the left of the origin. The state μ_θ may be contracted about θ; the resulting state is Poisson with an extra particle at the origin; therefore the previous results hold also for μ_θ.

Similar results hold for fields. We first note that (from the exact calculations of ref. 19, or computing using contractions as above)

$$\sup_{\varepsilon>0} EY_0^\varepsilon(\varphi)^2\leqslant C\,\|\varphi\|_2^2 \quad (2.46)$$

[$\|\cdot\|_2$ is the $L^2(\rho h(dv)\,dq)$-norm] for some constant C and all $\varphi\in\mathscr{S}(\mathbb{R}^2)$. By an argument identical to that used before [after Eq. (2.30)], one obtains from (2.45)

$$(Y_0^\varepsilon(\varphi_i))_{i=1}^n \xrightarrow{d} (Y_0(\varphi_i))_{i=1}^n \quad (2.47)$$

where $Y_0(\cdot)$ has covariance given in (2.18). Since tightness is also implied by (2.46), we have completed the proof of weak convergence of the time-zero fields. The convergence of the second moments (covariance) of the measure or field can also be proven with contractions, but we omit this as reproducing earlier work.[19] It is easy to check that the limiting Gaussian process given in (2.45) has the covariance given in (2.14). The error in the approximation is $O(\varepsilon^2)$ (for fields).

We turn next to the proof of the facts used concerning the dynamics. We contract about a "test" pulse (q_0, v_0). For a fixed hard-rod configuration X let $C_{q_0, X_t}(q)$ be the contracted position (at time t) in the time-evolved configuration X_t. $[C_{q_0, X_t}(q_0) = q_0 + O(1)$, the "$O(1)$" referring to a jump of at most $\pm d$ if q_0 is covered at time t.$]$ Let $n_t(q_0, v_0) \equiv$ algebraic number of crossings ("collisions") of the particle (q_0, v_0) in the contracted representation during the time interval $(0, t]$ in the free (ideal-gas) evolution. In terms of this collision number, the contracted image at time zero of the point (q, w) at time t is given by

$$x_t^c(q, w) = q - wt - dn_t(q_0, v_0) + O(1) \tag{2.48}$$

One must still "dilate" around $q_0 - v_0 t$ to obtain the hard-rod configuration at time 0. Equations such as (2.6) are easily checked in terms of the contracted dynamics.

We next prove the LLN for the asymptotic motion of the pulses. We consider the Palm measure at time t relative to the point (q_0, v_0), which can be thought of as the conditional measure, conditioned on a rod being there. For a test pulse the proof is essentially the same. Contracting around (q_0, v_0) at time t, we have

$$n_t(q_0, v_0) = n_t^+(q_0, v_0) - n_t^-(q_0, v_0) \tag{2.49}$$

where $n_t^\pm$ is the number of collisions from the right or left during the motion. In the contracted representation these are occupation numbers of the regions

$$\{(q', w) \colon \pm w < \pm v_0,\ \pm q_0 \geqslant \pm q',\ \pm[q' + (v_0 - w)t] \geqslant \pm q_0\} \tag{2.50}$$

The contracted state $C_{q_0}(X) \backslash (q_0, v_0)$ is Poisson on $\mathbb{R}^2$ with intensity $\rho_0\, dq\, h(dv)$. Hence

$$En_t^\pm(q_0, v_0) = \pm\rho t \int_{[\pm v_0, \pm\infty)} (v_0 - w)\, h(dw) \tag{2.51}$$

We conclude from the strong law of large numbers that

$$\varepsilon x_t^\varepsilon(q_0, v_0) - q_0 + v_0 t + d\rho_0 t \int_{-\infty}^{+\infty} (v_0 - v)\, h(dv)$$
$$= \varepsilon x_t^\varepsilon(q_0, v_0) + \bar{v}_0 t \to 0 \qquad \text{a.s.} \tag{2.52}$$

as $\varepsilon \to 0$.

We next prove Lemma 2.2. It is simplest to employ the exact calculation of the covariance (for fixed $\varepsilon > 0$) of ref. 16, employed also in ref. 19. Define

$$EY_t^\varepsilon(\varphi)\, Y_0^\varepsilon(\psi) \equiv (\psi, T_t^\varepsilon \varphi)_2 \tag{2.53}$$

where $(\ ,\ )_2$ is the inner product in $L^2(\rho\, dq\, h(dv))$. The previous authors found that in fact

$$T_t^\varepsilon = e^{A^\varepsilon t} T_0^\varepsilon \tag{2.54}$$

with A^ε the generator of a contraction semigroup. Since the underlying microscopic evolution is unitary (in L^2 of the equilibrium state), one has that

$$T_t^{\varepsilon *} = T_{-t}^\varepsilon \tag{2.55}$$

where the asterisk denotes adjoint in $L^2(\rho\, dq\, h(dv))$. Therefore

$$T_0^{\varepsilon *} = T_0^\varepsilon, \qquad A^\varepsilon T_0^\varepsilon = -T_0^\varepsilon (A^\varepsilon)^* \tag{2.56}$$

In fact, as shown in ref. 19, $A^\varepsilon = A_0 + \varepsilon A_1 + O(\varepsilon^2)$, where A_0 and A_1 are the Euler operator and Navier–Stokes correction, respectively.

Using (2.54) and expanding in t using Taylor's theorem and stationarity, one finds

$$E[Y_t^\varepsilon(\varphi) - Y_0^\varepsilon(\varphi)]^2 = 2[(\varphi, T_0^\varepsilon \varphi)_2 - (\varphi, e^{A^\varepsilon t} T_0^\varepsilon \varphi)_2]$$
$$= 2(\varphi, A^\varepsilon T_0^\varepsilon \varphi) t + 2 \int_0^t d\tau\, \tau(\varphi, (A^\varepsilon)^2\, T_\tau^\varepsilon \varphi)_2$$
$$\leqslant t^2 \sup_{\varepsilon > 0} \|(A^{\varepsilon *})^2\, \varphi\|_2^2 \leqslant C\, \|\varphi''\|_2^2\, t^2 \tag{2.57}$$

In (2.57) the term $O(t)$ vanishes by the skew-symmetry of A^ε [Eq. (2.56)]. The last bound follows from the exact calculations in ref. 19. This gives (2.29).

We next discuss how we can remove the restriction to a discrete velocity distribution. If the velocity distribution is continuous

$[h(dv) = h(v)\,dv]$, we must replace intervals in space ($\mathbb{R}$) by regions, rectangles say, in the one-particle phase space ($\mathbb{R}^2$). Accordingly, we define the fluctuation random measure in the continuous case by

$$Y_t^\varepsilon([a, b] \times [v_1, v_2])$$
$$\equiv \varepsilon^{1/2}[N_{\varepsilon^{-1}t}([\varepsilon^{-1}a, \varepsilon^{-1}b] \times [v_1, v_2])] - \rho\varepsilon^{-1}(b-a)\int_{v_1}^{v_2} h(w)\,dw \quad (2.58)$$

The first term on the right side contains the number of rods in the spatial interval $[\varepsilon^{-1}a, \varepsilon^{-1}b]$ with velocities between v_1 and v_2.

We next take $\delta > 0$ small and $V < \infty$ large. We divide phase space into horizontal strips of width δ and ignore velocities v with $|v| > V$. Assuming that v_1, v_2 are integer multiples of δ and writing

$$R^\varepsilon \equiv [\varepsilon^{-1}a, \varepsilon^{-1}b] \times [v_1, v_2]$$
$$= \bigcup_{j=k_1}^{k_2} [\varepsilon^{-1}a, \varepsilon^{-1}b] \times [(j-1)\delta, j\delta] \equiv \bigcup_{j=k_1}^{k_2} R_j^\varepsilon \quad (2.59)$$

we can then carry out the derivation leading to (2.12), approximating the random preimage at time zero of a rectangle R_j^ε by another rectangle, i.e., we write

$$N_{\varepsilon^{-1}t}(R_j^\varepsilon) = N_0([x_t^\varepsilon(a, j\delta), x_t^\varepsilon(b, j\delta)] \times [(j-1)\delta, j\delta]) + r_{j,\delta}^\varepsilon \quad (2.60)$$

The error $r_{j,\delta}^\varepsilon$ is incurred by replacing the random preimage of R_j^ε, which has irregular right and left boundaries, by a rectangle. Hence $r_{j,\delta}^\varepsilon = o(\delta)$ in the sense that

$$\lim_{\delta \to 0} \delta^{-1} \limsup_{\varepsilon \to 0} (E^\varepsilon r_{j,\delta}^\varepsilon) \vee [\varepsilon E(r_{j,\delta}^\varepsilon - Er_{j,\delta}^\varepsilon)^2]^{1/2} = 0$$

Another error is incurred by ignoring collisions with rods of velocity w, $|w| > V$. We choose $V = V(\delta)$ so large that $\delta^{-1}\int_{|v|>V} h(dv) \to 0$. Hence, if we define

$$R_{j,k}^\varepsilon = [x_t^\varepsilon(a, j\delta), x_t^\varepsilon(b, j\delta)] \times [(k-1)\delta, k\delta] \quad (2.61)$$

we have

$$Y_t^\varepsilon(R^\varepsilon) = \sum_j \sum_{k,l} C_{k\delta, j\delta} C_{l\delta, k\delta}^{-1} Y_0^\varepsilon(R_{k,l}^\varepsilon) + O(\delta) \quad (2.62)$$

Now Billingsley's theorem applies as before; with δ fixed, the first term on the right side of (2.62) therefore has a Gaussian limit (jointly for any finite number of times and rectangles). The covariance structure is still given by (2.14) and (2.16) [(v, w) replaced appropriately by $(j\delta, k\delta)$ and the intervals by strips of width δ].

We next pass to the limit $\delta \to 0$. It is necessary to switch to smooth test functions, so let $\phi \in C_0^\infty(\mathbb{R}^2)$. Approximating ϕ by linear combinations of characteristic functions of rectangles R^ε and using the argument given before [after (2.30)], we obtain, taking limits $\varepsilon \to 0$, the approximation of ϕ tending to ϕ, then $\delta \to 0$, a Gaussian limit of form

$$Y_t(\phi) = Y_0(T_t^* \phi) \tag{2.63}$$

with T_t^* given in (2.21) and Y_0 Gaussian with covariance

$$EY_0(\phi)\, Y_0(\psi) = \iint dv\, dw\, h(v)\, c(v, w) \int dq\, \phi(q, v)\, \psi(q, w) \tag{2.64}$$

and

$$c(v, w) = \delta(v - w) - \rho\, dh(v) \tag{2.65}$$

This gives Theorem 2.1 for continuous velocities. Theorem 2.2 follows as well, since tightness did not depend on having a discrete velocity distribution.

3. THE LLN AND NONEQUILIBRIUM FLUCTUATIONS ON THE EULER SCALE

In this section we compute the limit of the fluctuation random measure with Euler scaling and a nonequilibrium initial state (actually, a family of initial states). The limiting process is again deterministic and is governed by the linearized Euler equation, but now linearized around a nonstationary (time-dependent) solution. The method is the obvious generalization of the method of Section 2 (that is, we reduce the fluctuation measure at time t to a "random space change" of the measure at time zero, and then apply Billingsley's theorem). The formulas become somewhat more complicated. We confine ourselves to proving the appropriate generalization of Theorem 2.1. We use the setup of ref. 3; in particular, we will assume the regularity conditions from Theorem 4.1 there, as well as some other hypotheses on the family of initial states P^ε (we also show how the LLN follows from the results in ref. 3). The reader should first become familiar with the facts proven there.

Since the CLT can be regarded as describing the first corrections to the LLN, we discuss the LLN first. This follows readily (at least for discrete velocities) from (2.2), the results of ref. 3, and the (weak) LLN for the occupation numbers of random regions. The latter theorem asserts in our context that, for all $\delta > 0$,

$$P\left[\left|\varepsilon N_0([x_t^\varepsilon(a, v), x_t^\varepsilon(b, v)]; v) - \int_a^b \rho(q, t; v)\, dq\right| > \delta\right] \to 0 \qquad (3.1)$$

where $\rho(\cdot, \cdot)$ solves the Euler equation. This follows readily from (1) the weak LLN for the time-zero measure,

$$P\left[\varepsilon N_0([\varepsilon^{-1}a, \varepsilon^{-1}b]; v) - \int_a^b \rho(q, 0; v)\, dq > \delta\right] \to 0 \qquad (3.2)$$

and (2) the weak LLN for the collision numbers, with limiting value predicted by the "continuum analog" of the hard-rod fluid (see below). These are proven in ref. 3. The restriction to discrete velocities can be removed along the lines discussed in Section 2.

Let $\rho(q, v; t)$ be a solution of (1.8) satisfying the regularity conditions assumed in ref. 3, Section 3. Define, as in Section 2,

$$Y_t^\varepsilon([a, b]; v) = \varepsilon^{1/2}\left\{N_{\varepsilon^{-1}t}([\varepsilon^{-1}a, \varepsilon^{-1}b]; v) - \int_{\varepsilon^{-1}a}^{\varepsilon^{-1}b} \rho(\varepsilon q, v; t)\, dq\right\} \qquad (3.3)$$

Remarks. With this definition, $Y_\cdot^\varepsilon(\cdot)$ is *not* a centered random variable. The correct centering of the occupation number would be

$$\int_{\varepsilon^{-1}a}^{\varepsilon^{-1}b} k^{(1)}_{P^\varepsilon_{\varepsilon^{-1}t}}(q, v)\, dq \qquad (3.4)$$

where $k^1_{P_t^\varepsilon}$ is the first correlation function of P_t^ε.[3,4] We define $Y_\cdot^\varepsilon(\cdot)$ as in (3.3), since we are interested in the limiting fluctuations *around the Euler equation.* Since we shall prove that $Y_\cdot^\varepsilon(\cdot)$ has a mean-zero, Gaussian limit (with appropriate assumptions on P_0^ε), we shall establish indirectly that the difference between the second term in (3.3) and (3.4) goes to zero (with the prefactor $\varepsilon^{1/2}$). (That this difference is not of order $\varepsilon^{-1/2}$ is not surprising, since both quantities are averaged, i.e., nonrandom).

Proceeding as in Section 2, we rewrite (3.3) as

$$\begin{aligned} Y_t^\varepsilon([a, b]; v) = \varepsilon^{1/2}&\left\{N_0([x_t^\varepsilon(a, v), x_t^\varepsilon(b, v)], v) - \int_{x_t^\varepsilon(a, v)}^{x_t^\varepsilon(b, v)} \rho(\varepsilon q, v; 0)\, dq\right\} \\ &+ \varepsilon^{1/2}\left\{\int_{x_t^\varepsilon(a, v)}^{x_t^\varepsilon(b, v)} \rho(\varepsilon q, v; 0)\, dq - \int_{\varepsilon^{-1}a}^{\varepsilon^{-1}b} \rho(\varepsilon q, v; t)\, dq\right\} \\ &+ O(\varepsilon^{1/2}) \end{aligned} \qquad (3.5)$$

We write the terms in the first brackets in (3.5) as $Y_0^\varepsilon([x_t^\varepsilon(a, v), x_t^\varepsilon(b, v); v)$, as before; it will have a Gaussian limit by Billingsley's theorem. We proceed to treat the remaining terms.

Define $x_t(q, v)$ to be the "continuum analog" of the motion of the point (q, v) in an appropriate "continuum fluid" (ref. 3, Section 3). In the notation of ref. 3,

$$x_t(q, v) = u^{-1}_{\rho_0, v, t}(q), \qquad \rho_0(q, v) \equiv \rho(q, v; 0) \tag{3.6}$$

Since (ref. 3, Section 3)

$$\rho(q, v; t) = \rho_0(u^{-1}_{\rho_0, v, t}(q), v)(d/dq)\, u^{-1}_{\rho_0, v, t}(q) \tag{3.7}$$

we can write the last term in (3.5) after a change of variables as

$$\varepsilon^{-1} \int_{x_t(a, v)}^{x_t(b, v)} \rho(q, v; 0)\, dq \tag{3.8}$$

Make the change of variables $q \to \varepsilon^{1/2} q$ in the last bracket in (3.5). One obtains

$$\begin{aligned} &\int_0^{\varepsilon^{-1/2}[\varepsilon x_t^\varepsilon(b, v) - x_t(b, v)]} \rho_0(\varepsilon^{1/2} q + x_t(b, v), v)\, dq \\ &\qquad - \int_0^{\varepsilon^{-1/2}[\varepsilon x_t^\varepsilon(a, v) - x_t(a, v)]} \rho_0(\varepsilon^{1/2} q + x_t(a, v), v)\, dq \end{aligned} \tag{3.9}$$

Expanding the argument of ρ_0 and using the assumption that $d/dq\, \rho_0$ is uniformly bounded, we obtain for (3.9)

$$\begin{aligned} &\rho_0(x_t(b, v); v)\{\varepsilon^{-1/2}[\varepsilon x_t^\varepsilon(b, v) - x_t(b, v)]\} \\ &\qquad - \rho_0(x_t(a, v); v)\{\varepsilon^{-1/2}[\varepsilon x_t^\varepsilon(a, v) - x_t(a, v)]\} + O(\varepsilon^{1/2}) \end{aligned} \tag{3.10}$$

We treat the terms in (3.10) in a fashion similar to that in Section 2. We have

$$x_t^\varepsilon(q, v) = \varepsilon^{-1} q - \varepsilon^{-1} v t - d \sum_{w \neq v} N_0([x_t^\varepsilon(q, v), x_t^\varepsilon(q, w)]; w) + O(1) \tag{3.11}$$

so that, following the familiar route,

$$\begin{aligned} &\varepsilon^{1/2}[x_t^\varepsilon(q, v) - \varepsilon^{-1} x_t(q, v)] \\ &\quad = -d \sum_{w \neq v} Y_0^\varepsilon([x_t^\varepsilon(q, v), x_t^\varepsilon(q, w)]; w) \\ &\qquad - d \sum_{w \neq v} \int_{x_t^\varepsilon(q, w)}^{x_t^\varepsilon(q, v)} \rho(\varepsilon q', w; 0)\, dq' + \varepsilon^{-1} q - \varepsilon^{-1} v t \\ &\qquad - \varepsilon^{-1} x_t(q, v) + O(\varepsilon^{1/2}) \end{aligned}$$

$$= -d \sum_{w \neq v} Y_0^\varepsilon([x_t^\varepsilon(q, v), x_t^\varepsilon(q, w)]; w)$$
$$- d \sum_{w \neq v} \{\rho_0(x_t(q, v), w)\, \varepsilon^{1/2}[x_t^\varepsilon(q, v) - \varepsilon^{-1}x_t(q, v)]$$
$$- \rho_0(x_t(q, w), w)\, \varepsilon^{1/2}[x_t^\varepsilon(q, w) - \varepsilon^{-1}x_t(q, w)]\} + O(\varepsilon^{1/2}) \quad (3.12)$$

In deriving (3.12), we used the same change of variables and expansion used in deriving (3.10) and the following fact:

$$x_t(q, v) - q + vt + d \sum_{w \neq v} \int_{x_t(q, w)}^{x_t(q, v)} \rho_0(q', w)\, dq' \equiv 0 \quad (3.13)$$

Equation (3.13) is the analog of (2.6) and expresses the location at time zero of the volume of "fluid" with which our pulse collided during the motion in the time interval $[0, t]$. It is easily established from the explicit formulas in ref. 3.

Equation (3.12) can be regarded as an inhomogeneous system for $\xi(v) \equiv \varepsilon^{1/2}[x_t^\varepsilon(q, v) - \varepsilon^{-1}x_t(q, v)]$ of the form

$$\sum_w M_{v,w}\, \xi(w) = F(v) \quad (3.14)$$

where

$$M_{v,w} = M_{v,w}(q, t) \equiv [1 - d\rho_0(x_t(q, v), w)]\, \delta_{v,w} + d\rho_0(x_t(q, v), w)\, h(w) \quad (3.15)$$

Inverting,

$$\varepsilon^{1/2}[x_t^\varepsilon(q, v) - x_t(q, v)]$$
$$= \sum_w M_{v,w}^{-1}(q, t)(-d) \sum_{w' \neq w} Y_0^\varepsilon([x_t^\varepsilon(q, w), x_t^\varepsilon(q, w')]; w') \quad (3.16)$$

Combining these results, we arrive at

$$Y_t^\varepsilon([a, b]; v) = Y_0^\varepsilon([x_t^\varepsilon(a, v), x_t^\varepsilon(b, v)]; v)$$
$$- d\rho_0(x_t(b, v); v) \sum_w M_{v,w}^{-1}(b, t)$$
$$\times \sum_{w' \neq w} Y_0^\varepsilon([x_t^\varepsilon(b, w), x_t^\varepsilon(b, w')]; w')$$
$$+ d\rho_0(x_t(q, v); v) \sum_w M_{v,w}^{-1}(a, t)$$
$$\times Y_0^\varepsilon([x_t^\varepsilon(a, w), x_t^\varepsilon(q, w')]; w') + O(\varepsilon^{1/2}) \quad (3.17)$$

Equation (3.17) is the nonequilibrium analog of (2.12).

We shall need for the theorem (in addition to the hypotheses of ref. 3) that $Y_0^\varepsilon(\cdot) \to Y_0(\cdot)$ in distribution, where $Y_0(\cdot)$ is Gaussian with covariance

$$\begin{aligned} &EY_0(\varphi)\, Y_0(\psi) = (\varphi, C^2\psi)_2 \\ &[C\varphi](q, v) \equiv \varphi(q, v) - d \sum_w \rho(q, w)\, h(w)\, \varphi(q, w) \end{aligned} \tag{3.18}$$

Since the LLN for the asymptotic motion of the pulses,

$$\varepsilon x_t^\varepsilon(q, v) - x_t(q, v) \xrightarrow{P} 0 \tag{3.19}$$

is proven in ref. 3, we conclude from Billingsley's theorem that

$$\begin{aligned} Y_t^\varepsilon([a, b]; v) &\xrightarrow[\varepsilon \to 0]{} Y_t([a, b]; v) \\ &\equiv Y_0([x_t(a, v), x_t(b, v)]; v) - d\rho_0(x_t(b, v), v) \\ &\quad \times \sum_w M_{v,w}^{-1}(b, t) \sum_{w' \neq w} Y_0([x_t(b, w), x_t(b, w')]; w') \\ &\quad + d\rho_0(x_t(a, v), v) \sum_w M_{v,w}^{-1}(a, t) \\ &\quad \times \sum_{w' \neq w} Y_0([x_t(a, w), x_t(a, w')]; w') \end{aligned} \tag{3.20}$$

Finally, we have to check that $Y_t(\cdot)$ defined in (3.20) is the Gaussian process satisfying

$$Y_t(\varphi) = Y_0(u_{\rho_0}^*(t; 0)\varphi) \tag{3.21}$$

where $u_{\rho_0}(t; 0)$ is the solution operator of the linearized Euler equation, linearized around the nonstationary solution $\rho(q, t; v)$. In order to do so, we obtain an explicit form for the solution of the linearized equation in terms of ρ_0 and the $x_t(\cdot)$, and compare with (3.20).

It is convenient to start not with (1.8), but with the explicit expression (3.7). We write

$$\rho(q, t; v) = \tilde{\rho}(q, t; v) + \eta h(v)\, f(q, t; v)$$

and take $(d/d\eta)_0$ in (3.7). For convenience we let $\mathring{x}_t(q, v) \equiv (d/d\eta)_0\, x_t(q, v)$. We have

$$\begin{aligned} (d/d\eta)_0\, \rho(q, t; v) = h(v)\, f(x_t(q, v), 0; v)\, d/dq\, x_t(q, v) \\ + (d/dq\, \tilde{\rho}_0)(x_t(q, v), v)[d/dq\, x_t(q, v)]\, \mathring{x}_t(q, v) \\ + \tilde{\rho}_0(x_t(q, v), v)\, d/dq\, \mathring{x}_t(q, v) \end{aligned} \tag{3.22}$$

We derive an expression for the (functional) derivative $\mathring{x}_t(q, v)$ by taking $(d/d\eta)_0$ in (3.13), obtaining

$$\mathring{x}_t(q, v) = -d \sum_{w \neq v} \int_{x_t(q,w)}^{x_t(q,v)} f(q', 0; w')\, dq' - d \sum_{w \neq v} \{\rho_0(x_t(q, v))\, \mathring{x}_t(q, v) - \rho_0(x_t(q, w))\, \mathring{x}_t(q, w)\} \tag{3.23}$$

Inverting as usual, we obtain

$$\mathring{x}_t(q, v) = (-d) \sum_w M_{v,w}^{-1}(q, t) \sum_{w' \neq w} \int_{x_t(q,w')}^{x_t(q,w)} f(q', 0; w') \tag{3.24}$$

Thus, combining we obtain

$$\begin{aligned} f(q, t; v) = {} & f(x_t(q, v), 0; v)\, d/dq\, x_t(q, v) \\ & + [d/dq\, \tilde{\rho}_0(x_t(q, v), v)][d/dq\, x_t(q, v)] \sum_w M_{v,w}^{-1}(q, t) \\ & \times \sum_{w' \neq w} \int_{x_t(q,w')}^{x_t(q,w)} f(q', 0; w')\, dq' + \tilde{\rho}_0(x_t(q, v), v) \\ & \times d/dq \sum_w M_{v,w}^{-1}(q, t) \sum_{w' \neq w} \int_{x_t(q,w')}^{x_t(q,w)} f(q', 0; w')\, dq' \end{aligned} \tag{3.25}$$

This last expression gives explicitly the solution operator of the linearized equation, linearized around $\tilde{\rho}(q, t; v)$.

The hydrodynamic prediction for the structure of the nonequilibrium fluctuation process (on the Euler scale) is expressed by, symbolically,

$$Y_t([a, b]; v) \sim \int_a^b f(q, t; v)\, dq \tag{3.26}$$

Integrating over $[a, b]$ in (3.25), changing variables in the first term, and noting that the second plus third terms form a total derivative, one sees by comparing with (3.20) that this prediction is verified.

We summarize the hypotheses and conclusions in the following theorem.

Theorem 3.1. Let P^ε, $\varepsilon > 0$, be a family of hard-rod states satisfying the hypotheses of Theorem 4.1 of ref. 3 and the following:

(i) The density of the first correlation measure of P^ε, $k^1_{P^\varepsilon}(q, v)$, is of the form

$$k^1_{P^\varepsilon}(q, v) = \rho_0(\varepsilon q, v) \tag{3.27}$$

with ρ_0 satisfying the hypotheses of Theorem 3.1 of ref. 3 [$\rho_0(\cdot)$ and $d/dq\, \rho_0(\cdot)$ should be uniformly bounded and $d\rho_0(\cdot)$ should be uniformly bounded away from one].

(ii) The time-zero fluctuation measure

$$Y_0^\varepsilon([a, b]; v) = \varepsilon^{1/2} \left\{ N([\varepsilon^{-1}a, \varepsilon^{-1}b]; v) - \int_{\varepsilon^{-1}a}^{\varepsilon^{-1}b} \rho_0(\varepsilon q, v)\, dq \right\} \tag{3.28}$$

converges weakly in distribution to a Gaussian random measure $Y_0(\cdot)$ with covariance given in (3.18) (more precisely, the measure on $D((-\infty, \infty), \mathscr{S}'(\mathbb{R}^2))$ induced by $q \to Y_0^\varepsilon([0, q))$ converges weakly to that induced by $q \to Y_0([0, q))$).

Then the measure-valued process $Y_t^\varepsilon(\cdot)$ defined in (3.3) with $\rho(q, t; v)$ the (unique) solution of (1.8) with initial data $\rho(\cdot; 0) = \rho_0(\cdot)$ converges in distribution (jointly for any finite collection q_i, b_i, v_i, t_i) to the deterministic Gaussian process $Y_t(\cdot)$ satisfying

$$Y_t(\varphi) = Y_0(u(t; 0)\varphi) \tag{3.29}$$

where $u(t; 0)$ is the solution operator of the linearized Euler equation, linearized around $\rho(q, t; v)$.

Remark. Families P^ε satisfying (i) and the other hypotheses of that paper were constructed in ref. 3, Section 5. These were families of Gibbs states of short-range, one- and-two-body potentials. The Gaussian limit of the time-zero field for these measures [assumption (ii)] follows from well-known mixing conditions satisfied (uniformly as $\varepsilon \to 0$) by these measures, and from equally well known forms of the CLT for mixing random fields.[8]

APPENDIX. RANDOM SUMS OF RANDOM VARIABLES AND BILLINGSLEY'S THEOREM

We present a short discussion of Billingsley's theorem on the CLT for random sums of random variables, taken from ref. 1, Chapter 3, Section 12. For more details the reader should consult that reference.

Let $\xi_1, \xi_2, \ldots$ be jointly defined random variables with mean zero and finite variances, and assume the CLT holds: for every $t \geq 0$,

$$Y^\varepsilon(t) \equiv \varepsilon^{1/2} \sum_{i=0}^{[\varepsilon^{-1}t]} \xi_i \xrightarrow[\varepsilon \to 0]{d} Y(t) \tag{A.1}$$

where $Y(t)$ has a Gaussian distribution ($[\cdot] \equiv$ greatest integer function). Let $v^\varepsilon(t)$ be a positive r.v. defined jointly with the ξ and increasing a.s. Assuming that $v^\varepsilon(\cdot) \xrightarrow{d} v(\cdot)$ (in a suitable sense) for some r.v. $v(\cdot)$ defined jointly with $Y(\cdot)$, we inquire whether or not one can conclude that

$$Y^\varepsilon(v^\varepsilon(t)) \xrightarrow[\varepsilon \to 0]{d} Y(v(t)) \tag{A.2}$$

Thus, we are looking for a limit theorem for a random sum of random variables. We have set up the problem as a "random time change," although in this paper the corresponding problem appears as a "random space change," at fixed times.

Billingsley's theorem holds with the following hypotheses: The *joint* distribution P^ε on $D([0, \infty); \mathbb{R}^2)$ induced by $t \to (Y^\varepsilon(t), v^\varepsilon(t))$ converges weakly to that induced by $(Y(t), v(t))$, and in addition $Y(\cdot)$ and $v(\cdot)$ are continuous a.s. The proof exploits the continuity property of weak convergence under the composition map: $(Y(\cdot), v(\cdot)) \to Y \circ v(\cdot)$. One concludes from the theorem the joint convergence

$$(Y^\varepsilon(v^\varepsilon(t_i)))_{i=1}^n \xrightarrow{d} (Y(v(t_i)))_{i=1}^n$$

for any choice of $t_1,..., t_n$ (convergence of the finite-dimensional marginals), although the actual conclusion of the theorem (weak convergence) is considerably stronger.

An important special case of Billingsley's theorem (used in this paper to derive the Euler limit of the fluctuation random measure) is for $Y^\varepsilon(\cdot) \to$ Gaussian process and $v^\varepsilon(t) \to$ (constant) t in probability.

Billingsley includes in the statement of his theorem an additional hypothesis: that the "time-change" function remains bounded. However, in applying Billingsley's theorem, we usually do not have to impose this condition; for, in the definition of weak convergence, we use only bounded functions, and thus it suffices to know that

$$\lim_{M \to \infty} \sup_{\varepsilon > 0} P[\sup_{0 \leqslant \tau \leqslant T} v_\varepsilon(\tau) > M] = 0 \tag{A.3}$$

e.g., in this paper,

$$\lim_{M \to \infty} \sup_{\varepsilon} P[\sup_{a \leqslant q \leqslant b} |\varepsilon x_{\varepsilon^{-1}t}(q, v)| > M] = 0 \tag{A.4}$$

which follows readily using contractions.

We note that many expressions in this paper appear first as random sums of *noncentered* random variables, and the first step is always to center the sum properly (by subtracting a *random* centering term). For instance, if $\eta_1, \eta_2,\ldots$ forms a stationary sequence with $E\eta_i = E\eta_1 = m \neq 0$, one centers as

$$\varepsilon^{1/2}\left\{\sum_{i=1}^{[\varepsilon^{-1}v^\varepsilon(t)]} \eta_i - \varepsilon^{-1}v^\varepsilon(t)m\right\} \tag{A.5}$$

instead of [assuming $v^\varepsilon(t) \sim t$]

$$\varepsilon^{1/2}\left\{\sum_{i=1}^{[v^\varepsilon(t)]} \eta_i - \varepsilon^{-1}tm\right\} \tag{A.6}$$

The difference between (A.3) and (A.4),

$$\varepsilon^{1/2}\{\varepsilon^{-1}v^\varepsilon(t) - \varepsilon^{-1}t\}m \tag{A.7}$$

may make an additional contribution if $v^\varepsilon(t)$ has normal fluctuations. For instance, if the η_i are i.i.d. with variance 1, and $v^\varepsilon(t) \to t$ a.s., then (A.5) will have as limit a standard Brownian motion $B(t)$ $[EB(t)^2 = t]$, while (A.6), if $\varepsilon^{-1}v^\varepsilon(t)$ has normal fluctuations, will yield a Brownian motion with a different variance.

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